Kendall’s 1974 derivation of the cosine quantogram as a method for determining the existence of as well as deriving an estimate for a quantum unit is based on the following:

Suppose we wish to demonstrate that an unknown quantum unit $q$ has given rise to our data, where $Y_i$ represents the measurement of object $i$, $i=1,...,N$. Specifically we hypothesize that the $i$th observation, $Y_i$, is an unknown number of integer multiples $M_i$ of our unknown quantum unit plus a small amount of unknown error. Consider for a moment a simple model $Y_i = M_i q + \epsilon_i$, $i = 1, ..., N$. For a fixed $q$, if $q$ is a quantum, the associated errors should be small. For $q$ over some range $(q_L, q_U)$, calculate

$$\phi(q) = 2N \frac{1}{N} \sum_{i=1}^{N} \cos\left(\frac{2\pi x_i}{q}\right)$$

Graph $\phi(q)$ vs. $q$; call this the cosine quantogram.

We wish to find $q^*$ such that $\phi(q)$ is maximized. This will be our maximum likelihood estimate. Note that cosine quantogram is proportional to the log-likelihood for a von Mises distribution with density function

$$f(x \mid \delta, q) = \left(2\pi I_0(\delta)\right)^{-1} e^{\delta \cos(2\pi x / q)}$$

where $\delta \geq 0$, $x > 0$, $q > 0$ and $I_0(\delta)$ is a modified Bessel function of order 0. The maximum is determined numerically for some $q$ in the interval $(q_L, q_U)$. Prior to undertaking this numerical procedure, however, Kendall recommends “unrounding the data”. The process of unrounding involves adding a random value within $(-0.05, 0.05)$ for measurements rounded to the nearest decimeter, a random value within $(-0.005, 0.005)$ for measurements rounded to the nearest centimeter, and so on.

Our Approach: A Generalized Framework for Quantal Inference

We have developed a generalized framework for inference for quanta as follows:

Let $Y_{ijkl}$ represent the measurement of block $l$ from building $i$ of type $j$ with measurement taken along dimension $k$. In general we can express the model for quantum identification as follows:

$$Y_{ijkl} = M_{ijkl} q_{ijk} + \epsilon_{ijkl},$$

where $M_{ijkl}$ is an unknown positive integer, $q_{ijk}$ is the unknown quantum that we wish to estimate, and $\epsilon_{ijkl}$ is an error term.

We can use this framework now to construct hypotheses regarding the $q$'s, then build test statistics on the ratio of the likelihood with respect to one hypothesis vs. a suitable alternative. Likelihood theory is that the maxima of the relevant cosine quantograms can be used to construct this test. In particular we wish to test the hypothesis that the quantum units used for all buildings in our sample are the same. We write this hypothesis symbolically as a null hypothesis $H_0: q_i = q$, $\forall i$, where we use the “dot” notation to indicate that the factor that is represented with ‘.’ is averaged over in the analysis. For inferential purposes we write our alternative hypothesis as $H_A: q_i \neq q_i$, for at least one value of $i$. We note that our inferential method will allow for testing the identity of the quantum units used in 3 or more buildings as well.

Our method then proceeds in stepwise fashion as follows:

Step 1: Unround the analytic dataset.
Step 2: Derive cosine quantogram and numerically estimate the quantum unit. Call this $q^*$.
Step 3: Randomly resample from the empirical distribution function of measurements. Repeat Step 2.
Step 4: Repeat Step 3 a large number of times. Obtain confidence limit estimates from the empirical distribution function of the estimates obtained.
Step 5: For the unrounded data from Step 1, calculate the likelihood of the data under $H_0$ and under $H_A$. Take the ratio of these likelihoods (or alternatively, the difference between the log-likelihoods). Call this the test statistic.

Step 6: For the random resamples generated in step 4, calculate the likelihoods and the difference between the log-likelihoods.

Step 7: Obtain the empirical distribution function of these log-likelihoods.

Step 8: From this empirical distribution function, derive a critical region for a test of size $\alpha$.

Step 9: Determine if the test statistic falls inside or outside of the critical region, and reject or not reject $H_0$ in favor of $H_A$ accordingly.

If we assume that the data follow a von Mises distribution, we use as our test statistic the logarithm of likelihood ratio (LLR) written as follows:

$$LLR = \sum \sum \sum \sum \delta \cos(2\pi y_{ijkl} / \hat{q}_{ijkl}) - \sum \sum \sum \sum \delta \cos(2\pi y_{ijkl} / \hat{q}_{ijkl}^*) + c(\bar{N})$$

where $c(\bar{N})$ is a constant function dependent only on $\bar{N}$, the vector of the sample sizes of units for all possible combinations of the indices. Here the superscripts *0 and *A are used to denote under which hypothesis the quantal value(s) and resulting error terms are estimated using the cosine quantogram. Let $T_1$ designate the term in the LLR and $T_2$ the second term. Since the constant will not matter in our testing, our test statistic $T_{\text{diff}} = T_1 - T_2$. However note that $T_{\text{diff}}$ is also a function of the parameter $\delta$. Still, we can factor out $\delta$ and use as test statistic $T_{\text{diff}}/\delta$ without loss of power. We can resample the dataset to arrive at point and interval estimates for $q_{ijkl}$ as well as to derive test boundaries for $T_{\text{diff}}/\delta$. We note that we can develop $T_{\text{diff}}/\delta$'s for any hypotheses, use the cosine quantograms to estimate the $q$'s and perform our hypothesis test.